

A Superstring Theory in $2 + 2$ DimensionsZ. Khviengia, H. Lu¹, C.N. Pope^{1,2}, E. Sezgin³, X.J. Wang and K.W. Xu

Center for Theoretical Physics, Texas A&M University, College Station, TX 77843-4242

Abstract

In this paper we construct a $(2, 2)$ dimensional string theory with manifest $N = 1$ space-time supersymmetry. We use Berkovits' approach of augmenting the spacetime supercoordinates by the conjugate momenta for the fermionic variables. The worldsheet symmetry algebra is a twisted and truncated "small" $N = 4$ superconformal algebra. The physical spectrum of the open string contains an infinite number of massless states, including a supermultiplet of a self-dual Yang-Mills field and a right-handed spinor and a supermultiplet of an anti-self-dual Yang-Mills field and a left-handed spinor. The higher-spin multiplets are natural generalisations of these self-dual and anti-self-dual multiplets.

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1 Introduction

The Green-Schwarz superstring [1], with manifest spacetime supersymmetry, has proved to be notoriously difficult to quantise in a covariant manner. The difficulty stems from the fact that there is no kinetic term for the fermionic spacetime coordinates. This problem has been overcome recently by Berkovits [2] in a reformulation of the superstring, in which the spacetime supercoordinates are augmented by the conjugate momenta for the fermionic variables. The theory has $N = 2$ worldsheet supersymmetry, as well as manifest four-dimensional $N = 1$ spacetime supersymmetry. This theory can be thought of as a ten-dimensional theory compactified on a Calabi-Yau background.

It is interesting to investigate whether such an approach could be used for constructing an intrinsically four-dimensional theory with manifest spacetime supersymmetry. This would contrast strikingly with the $N = 2$ NSR string [3] which, although it has a four-dimensional spacetime (with $(2, 2)$ signature), has no supersymmetry in spacetime. In fact it has only one physical state, describing self-dual Yang-Mills in the open string, and self-dual gravity in the closed string. Attempts have been made to find a supersymmetric version of the theory. In a recent paper [4], it was observed that massless fermionic physical states, as well as bosonic ones, appear in certain Z_2 twisted sectors of the theory. There is a kind of twisted $N = 2$ spacetime supersymmetry, with two chiral supercharges whose anticommutator vanishes.

If a four-dimensional string of the Berkovits type could be constructed, it would be quite different from the above case, in that it would have a manifest spacetime supersymmetry. A way to build such a theory is suggested by some work of Siegel [5]. He considered a set of quadratic constraints built from the coordinates and momenta of a superspace in $2 + 2$ dimensions, and thus displaying manifest spacetime supersymmetry. In [5] it was proposed that in the case of open strings, this theory described self-dual $N = 4$ super Yang-Mills, whilst the corresponding closed string described self-dual $N = 8$ supergravity.

The full set of constraints considered in [5] do not generate a closed algebra. However, we find that there exists a subset of the constraints that does close on an algebra, with two bosonic spin-2 generators and two fermionic spin-2 generators. In this paper we build a Berkovits-type open string theory in four dimensions, based on this worldsheet symmetry algebra. Noting that the central charge in the ghost sector vanishes, we see that the matter fields should also have zero central charge. We achieve this by taking the coordinates (X^μ, θ^α) of a chiral $N = 1$ superspace, together with the canonical momenta p_α for the fermionic coordinates. This is a chiral restriction of the analogous matter system introduced by Berkovits [2].

The chiral truncation that we are making is possible only if the signature of the four-dimensional spacetime is $(2, 2)$. In this case, the $SO(2, 2)$ Lorentz group is the direct product $SL(2, R)_L \times SL(2, R)_R$, with dotted spinorial indices transforming under $SL(2, R)_L$, and undotted indices transforming under $SL(2, R)_R$. The bosonic currents are singlets under the entire Lorentz group, but the two fermionic currents form a doublet under $SL(2, R)_L$. This means, as we shall show later, that while the physical states of the theory will have manifest $SL(2, R)_R$ spacetime symmetry, the $SL(2, R)_L$ symmetry, although remaining unbroken, is not always manifest.

The spectrum of physical states of this $(2, 2)$ dimensional theory turns out to be quite rich. Owing to the fact that the fermionic currents, and hence their associated ghosts, carry $SL(2, R)_L$ indices it follows that the ghost vacuum can transform non-trivially under $SL(2, R)_L$. As a result, it turns out that the physical spectrum includes an infinite number of massless

states, with arbitrarily high spin. Included amongst these is a supermultiplet comprising an anti-self-dual Yang-Mills field and a left-handed spinor. There is also supermultiplet consisting of a self-dual Yang-Mills field and a right-handed spinor. The higher-spin multiplets are natural generalisations of these anti-self-dual and self-dual multiplets. Such higher-spin massless fields seem not to have been encountered previously in string theory.

2 The constraint algebra and BRST charge

In this section, we set up the algebra of constraints that defines the string theory, and construct the BRST operator. The matter system consists of the four spacetime coordinates $X^{\alpha\dot{\alpha}} = \sigma_{\mu}^{\alpha\dot{\alpha}} X^{\mu}$, the two-component Majorana-Weyl spinor θ^{α} , and its conjugate momentum p_{α} . In the language of conformal field theory, these satisfy the OPEs

$$X^{\alpha\dot{\alpha}}(z)X^{\beta\dot{\beta}}(w) \sim -\epsilon^{\alpha\beta}\epsilon^{\dot{\alpha}\dot{\beta}}\log(z-w), \quad p_{\alpha}(z)\theta_{\beta}(w) \sim \frac{\epsilon_{\alpha\beta}}{z-w}. \quad (1)$$

When we need to be explicit, we use conventions in which the spacetime metric is given by $\eta_{\mu\nu} = \text{diag}(-1, -1, 1, 1)$, the indices $\mu, \nu \dots$ run from 1 to 4, and the mapping between tensor indices and 2-component spinor indices is defined by

$$V^{\alpha\dot{\alpha}} = \begin{pmatrix} V^{1\dot{1}} & V^{1\dot{2}} \\ V^{2\dot{1}} & V^{2\dot{2}} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} V^1 + V^4 & V^2 - V^3 \\ V^2 + V^3 & -V^1 + V^4 \end{pmatrix}, \quad (2)$$

where V^{μ} is an arbitrary vector. The Van der Waerden symbols $\sigma_{\mu}^{\alpha\dot{\alpha}}$ thus defined satisfy $\sigma_{\mu}^{\alpha\dot{\alpha}}\sigma^{\mu\beta\dot{\beta}} = \epsilon^{\alpha\beta}\epsilon^{\dot{\alpha}\dot{\beta}}$, where $\epsilon_{12} = \epsilon^{12} = 1$. Spinor indices are raised and lowered according to the usual “North-west/South-east” convention, with $\psi^{\alpha} = \epsilon^{\alpha\beta}\psi_{\beta}$ and $\psi_{\alpha} = \psi^{\beta}\epsilon_{\beta\alpha}$, *etc.*, so we have $\psi^1 = \psi_2$ and $\psi^2 = -\psi_1$. Note that since the indices are two-dimensional, we have the useful Schoutens identity $X^{\alpha}Y_{\alpha}Z_{\beta} + X_{\alpha}Y_{\beta}Z^{\alpha} + X_{\beta}Y^{\alpha}Z_{\alpha} = 0$.

In [5], Siegel proposed to build a string theory implementing the set of constraints given by $\{\partial X^{\alpha\dot{\alpha}}\partial X_{\alpha\dot{\alpha}}, p_{\alpha}\partial\theta^{\alpha}, p_{\alpha}p^{\alpha}, \partial\theta_{\alpha}\partial\theta^{\alpha}, p_{\alpha}\partial X^{\alpha\dot{\alpha}}, \partial\theta_{\alpha}\partial X^{\alpha\dot{\alpha}}\}$. However, it follows from (1) that whilst the second order poles in the OPEs amongst this set of constraints give back the same set of constraints, not all the first-order poles can be re-expressed as the derivatives of the constraints. In other words, the algebra does not close. (Note that this non-closure occurs even at the classical level of Poisson brackets, or single OPE contractions.) Accordingly, we choose a subset of Siegel’s constraints that form a closed algebra, namely

$$\begin{aligned} T &= -\frac{1}{2}\partial X^{\alpha\dot{\alpha}}\partial X_{\alpha\dot{\alpha}} - p_{\alpha}\partial\theta^{\alpha}, \\ S &= -p_{\alpha}p^{\alpha}, \\ G^{\dot{\alpha}} &= -p_{\alpha}\partial X^{\alpha\dot{\alpha}}. \end{aligned} \quad (3)$$

We see from the energy-momentum tensor that from the worldsheet viewpoint, θ^{α} has conformal weight 0, and p_{α} has conformal weight 1. Thus the two bosonic currents T and S , and also the two fermionic currents $G^{\dot{\alpha}}$, have conformal spin 2. The currents are all primary, and the remaining non-trivial OPE is given by

$$G^{\dot{\alpha}}(z)G^{\dot{\beta}}(w) \sim \frac{2\epsilon^{\dot{\alpha}\dot{\beta}}S}{(z-w)^2} + \frac{\epsilon^{\dot{\alpha}\dot{\beta}}\partial S}{z-w}. \quad (4)$$

The action for the matter system takes the form

$$I = \int d^2 z \left(-\frac{1}{2} \partial X^{\alpha\dot{\alpha}} \bar{\partial} X_{\alpha\dot{\alpha}} + p_{\alpha} \bar{\partial} \theta^{\alpha} \right). \quad (5)$$

Turning now to the BRST operator, we begin by introducing the anticommuting ghosts (b, c) and (β, γ) for the bosonic currents T and S ; and the commuting ghosts $(r^{\dot{\alpha}}, s_{\dot{\alpha}})$ for the fermionic currents $G^{\dot{\alpha}}$, with $r^{\dot{\alpha}}(z) s_{\dot{\beta}}(w) \sim -\epsilon^{\dot{\alpha}\dot{\beta}} (z-w)^{-1}$. All anti-ghosts $(b, \beta, r^{\dot{\alpha}})$ have spin 2, and all ghosts $(c, \gamma, s_{\dot{\alpha}})$ have spin -1 . Straightforward computation leads to the following result for the BRST operator:

$$Q = Q_0 + Q_1 + Q_2, \quad (6)$$

$$Q_0 = \oint c \left(-\frac{1}{2} \partial X^{\alpha\dot{\alpha}} \partial X_{\alpha\dot{\alpha}} - p_{\alpha} \partial \theta^{\alpha} - b \partial c - 2\beta \partial \gamma - \partial \beta \gamma + 2r^{\dot{\alpha}} \partial s_{\dot{\alpha}} + \partial r^{\dot{\alpha}} s_{\dot{\alpha}} \right), \quad (7)$$

$$Q_1 = \frac{1}{2} \oint \gamma p_{\alpha} p^{\alpha}, \quad (8)$$

$$Q_2 = \oint (s_{\dot{\alpha}} G^{\dot{\alpha}} - \beta s_{\dot{\alpha}} \partial s^{\dot{\alpha}}). \quad (9)$$

(Note that we can augment our currents (3) by including $\{\theta_{\alpha} \theta^{\alpha}, p_{\alpha} \theta^{\alpha}, \theta_{\alpha} \partial X^{\alpha\dot{\alpha}}\}$ as well. One can easily verify that the resulting currents generate precisely the “small” $N = 4$ superconformal algebra, but in a twisted basis. If the currents are then untwisted, it can easily be seen that the algebra has central charge $c = 6$, and therefore this realisation could not be used to build a nilpotent $N = 4$ BRST operator, which would require $c = -12$ for criticality. Thus one cannot build another string theory with more constraints just by enlarging our set (3) to give a full $N = 4$ superconformal set of currents.⁴)

Under $SL(2, R)_{\text{R}}$ transformations, corresponding to the self-dual Lorentz transformations of the undotted indices, the currents are all invariant. Thus the generators of $SL(2, R)_{\text{R}}$ are simply given by

$$J^{\alpha\beta} = \oint \left(-\frac{1}{2} X^{(\alpha}{}_{\dot{\alpha}} \partial X^{\beta)\dot{\alpha}} + p^{(\alpha} \theta^{\beta)} \right), \quad (10)$$

and, for an infinitesimal transformation with (symmetric) parameter $\omega_{\alpha\beta}$, have the action $\delta\psi^{\alpha} = [\omega_{\beta\gamma} J^{\beta\gamma}, \psi^{\alpha}] = \omega^{\alpha}{}_{\beta} \psi^{\beta}$ on any undotted index. The $SL(2, R)_{\text{L}}$ transformations, on the other hand, which correspond to anti-self-dual Lorentz rotations of the dotted indices, rotate the fermionic currents $G^{\dot{\alpha}}$, and hence the ghosts $(r^{\dot{\alpha}}, s_{\dot{\alpha}})$ must rotate also. It is quite easy to see that the generators are given by

$$J^{\dot{\alpha}\dot{\beta}} = \oint \left(\frac{1}{2} X^{\alpha(\dot{\alpha}} \partial X_{\alpha}{}^{\dot{\beta})} + r^{(\dot{\alpha}} s^{\dot{\beta})} \right), \quad (11)$$

and they transform dotted indices according to $\delta\psi^{\dot{\alpha}} = \omega^{\dot{\alpha}}{}_{\dot{\beta}} \psi^{\dot{\beta}}$. These spacetime Lorentz transformations are symmetries of the two-dimensional action including ghosts, and they commute with the BRST charge.

⁴ The $N = 4$ superconformal algebra, and its twisted subalgebra generated by (3), have been used recently in the construction of a topological $N = 4$ string [6]. This is a different kind of string from the one we are discussing, with no ghosts, and a BRST operator given by a fermionic spin-1 current in the twisted algebra which was one of the four spin- $\frac{3}{2}$ currents prior to twisting.

It is also useful to write down the form of the spacetime supersymmetry generators. These are given by

$$q_\alpha = \oint p_\alpha , \quad (12)$$

$$q^{\dot{\alpha}} = \oint \left(-\theta_\alpha \partial X^{\alpha\dot{\alpha}} - \gamma r^{\dot{\alpha}} + b s^{\dot{\alpha}} \right) . \quad (13)$$

The somewhat unusual ghost terms in $q^{\dot{\alpha}}$ are a consequence of the fact that $r^{\dot{\alpha}}$ and $s_{\dot{\alpha}}$ transform under the anti-self-dual spacetime Lorentz group. It is straightforward to verify using (1) that these supercharges generate the usual $N = 1$ spacetime superalgebra

$$\{q_\alpha, q_\beta\} = 0 = \{q^{\dot{\alpha}}, q^{\dot{\beta}}\}, \quad \{q^\alpha, q^{\dot{\alpha}}\} = P^{\alpha\dot{\alpha}} , \quad (14)$$

where $P^{\alpha\dot{\alpha}} = \oint \partial X^{\alpha\dot{\alpha}}$ is the spacetime translation operator.

As usual in a theory with fermionic currents, it is appropriate to bosonise the associated commuting ghosts. Thus we write

$$r^{\dot{\alpha}} = \partial \xi^{\dot{\alpha}} e^{-\phi_{\dot{\alpha}}} , \quad s_{\dot{\alpha}} = \eta_{\dot{\alpha}} e^{\phi_{\dot{\alpha}}} , \quad (15)$$

where $\eta_{\dot{\alpha}}$ and $\xi^{\dot{\alpha}}$ are anticommuting fields with spins 1 and 0 respectively. The OPEs of the bosonising fields are $\eta_{\dot{\alpha}}(z)\xi^{\dot{\beta}}(w) \sim \delta_{\dot{\alpha}}^{\dot{\beta}}(z-w)^{-1}$, and $\phi_{\dot{\alpha}}(z)\phi_{\dot{\beta}}(w) \sim -\delta_{\dot{\alpha}\dot{\beta}} \log(z-w)$. Note that the bosonisation breaks the manifest $SL(2, R)_L$ covariance, and that the $\dot{\alpha}$ index in (15) is not summed. In view of this non-covariance, there is no particular advantage in using upper as well as lower indices on $\phi_{\dot{\alpha}}$, and we find it more convenient always to use lower ones for this purpose.

The BRST operator is easily re-expressed in terms of the bosonised fields; the (r, s) terms in Q_0 become $\oint c \left(-\eta_{\dot{\alpha}} \partial \xi^{\dot{\alpha}} - \frac{1}{2}(\partial \phi_1)^2 - \frac{1}{2}(\partial \phi_2)^2 - \frac{3}{2}\partial^2 \phi_1 - \frac{3}{2}\partial^2 \phi_2 \right)$, whilst Q_2 becomes

$$\begin{aligned} Q_2 &= \oint \left(\eta_1 p_\alpha \partial X^{\alpha 1} e^{\phi_1} + \eta_2 p_\alpha \partial X^{\alpha 2} e^{\phi_2} \right) \\ &+ \oint \beta \left(\eta_1 \partial \eta_2 - \partial \eta_1 \eta_2 - \eta_1 \eta_2 (\partial \phi_1 - \partial \phi_2) \right) e^{\phi_1 + \phi_2} . \end{aligned} \quad (16)$$

(It is to be understood that an expression such as $e^{\phi_1 + \phi_2}$ really means $: e^{\phi_1} :: e^{\phi_2} :$, which equals $- : e^{\phi_2} :: e^{\phi_1} :$ since both of these exponentials are fermions. Thus we have $e^{\phi_1 + \phi_2} = -e^{\phi_2 + \phi_1}$ in this rather elliptical notation.)

The ghost contributions to the $SL(2, R)_L$ Lorentz generators (11) become

$$\begin{aligned} J_+ &= r_1 s_1 = \eta_1 \partial \xi^2 e^{\phi_1 - \phi_2} , \\ J_- &= r_2 s_2 = \eta_2 \partial \xi^1 e^{-\phi_1 + \phi_2} , \end{aligned} \quad (17)$$

$$J_3 = r_{(1} s_2) = -\frac{1}{2}(\partial \phi_1 - \partial \phi_2) . \quad (18)$$

One can easily see that these generate an $SL(2, R)$ Kac-Moody algebra. The translation of the supersymmetry charges into bosonised form is obtained by simple substitution.

Since the zero modes of the $\xi^{\dot{\alpha}}$ fields are not included in the Hilbert space of physical states, there exist BRST non-trivial picture-changing operators $Z^{\dot{\alpha}} = \{Q, \xi^{\dot{\alpha}}\}$ which can give new

BRST non-trivial physical operators when normal ordered with others. Explicitly, they take the form

$$Z^1 = c \partial \xi^1 - p_\alpha \partial X^{\alpha 1} e^{\phi_1} - \left(2\beta \partial \eta_2 + \partial \beta \eta_2 + 2\beta \eta_2 \partial \phi_2 \right) e^{\phi_1 + \phi_2} , \quad (19)$$

$$Z^2 = c \partial \xi^2 - p_\alpha \partial X^{\alpha 2} e^{\phi_2} - \left(2\beta \partial \eta_1 + \partial \beta \eta_1 + 2\beta \eta_1 \partial \phi_1 \right) e^{\phi_1 + \phi_2} , \quad (20)$$

Unlike the picture-changing operator in the usual $N = 1$ NSR superstring, it appears that here the operators have no inverse. This is similar to the situation in the $N = 1$ superstring formulation of [2].

3 Physical states

3.1 Preliminaries

In this section, we shall discuss the cohomology of the BRST operator, and present some results for physical states in the theory. Owing to the rather unusual feature in this theory that some of the ghosts carry target spacetime spinor indices, the notion of the standard ghost vacuum requires some modification. We begin by noting that the non-vanishing correlation function that defines the meaning of conjugation is given by

$$\left\langle \partial^2 c \partial c c \partial^2 \gamma \partial \gamma \gamma e^{-3\phi_1 - 3\phi_2} \theta^2 \right\rangle \neq 0 , \quad (21)$$

where $\theta^2 \equiv \theta^\alpha \theta_\alpha$. In terms of the bosonised form of the commuting ghosts, the usual operator $e^{-\phi_1 - \phi_2}$ appearing in the definition of the ghost vacuum can be generalised to an operator $W_{\dot{\alpha}_1 \dots \dot{\alpha}_{2s}}$, totally symmetric in its indices, whose component with $(s+m)$ indices taking the value $\dot{1}$ and $(s-m)$ taking the value $\dot{2}$ is given by

$$W_{\dot{1} \dots \dot{1} \dot{2} \dots \dot{2}} = \lambda(s, m) \partial^{s+m-1} \eta_1 \dots \partial \eta_1 \eta_1 \partial^{s-m-1} \eta_2 \dots \partial \eta_2 \eta_2 e^{(s+m-1)\phi_1 + (s-m-1)\phi_2} . \quad (22)$$

The normalisation constants $\lambda(s, m)$ are given by

$$\lambda(s, m) = \prod_{p=1}^{s+m-1} \prod_{q=1}^{s-m-1} \frac{1}{p! q!} , \quad (23)$$

where any product over an empty range is defined to be 1. W in (22) has $(s+m)$ factors involving η_1 , and $(s-m)$ factors involving η_2 , with $-s \leq m \leq s$. It is the $J_3 = m$ state in the $(2s+1)$ -dimensional spin- s representation of $SL(2, R)_L$. The operator $W_{\dot{1} \dots \dot{1}}$ corresponds to the highest-weight state in the representation, satisfying $J_+ W_{\dot{1} \dots \dot{1}} = 0$, with the remaining $2s$ states being obtained by acting repeatedly with J_- , each application of which turns a further “ $\dot{1}$ ” index into a “ $\dot{2}$ ”, until the lowest-weight state $W_{\dot{2} \dots \dot{2}}$ is obtained. Note, incidentally, that the form of the states given in (22) becomes rather simple if one bosonises the (η, ξ) fields.

We may also define a “conjugate” operator $\widetilde{W}^{\dot{\alpha}_1 \dots \dot{\alpha}_{2s}}$, again totally symmetric in its indices, by

$$\widetilde{W}^{\dot{1} \dots \dot{1} \dot{2} \dots \dot{2}} = \tilde{\lambda}(s, m) \partial^{s+m} \xi^1 \dots \partial^2 \xi^1 \partial \xi^1 \partial^{s-m} \xi^2 \dots \partial^2 \xi^2 \partial \xi^2 e^{-(s+m+2)\phi_1 - (s-m+2)\phi_2} , \quad (24)$$

with

$$\tilde{\lambda}(s, m) = \prod_{p=1}^{s+m} \prod_{q=1}^{s-m} \frac{1}{p! q!} . \quad (25)$$

Thus the usual ghost vacuum operator $e^{-\phi_1-\phi_2}$ and its “conjugate” $e^{-2\phi_1-2\phi_2}$ correspond to the $s = 0$ cases W and \widetilde{W} respectively. All the operators $W_{\dot{\alpha}_1 \dots \dot{\alpha}_{2s}}$ and $\widetilde{W}^{\dot{\alpha}_1 \dots \dot{\alpha}_{2s}}$ have worldsheet conformal spin 2, and they all have the property of defining vacuum states that are annihilated by the positive Laurent modes of $r^{\dot{\alpha}}$ and $s_{\dot{\alpha}}$, but not by the negative modes.

These operators have simple properties when acted on by the BRST operator. The relevant facts can be summarised in the following lemmas:

$$Q_2 W_{\dot{\alpha}_1 \dots \dot{\alpha}_{2s}} \theta^\alpha e^{ip \cdot X} = ip^{\alpha \dot{\alpha}_{2s+1}} W_{\dot{\alpha}_1 \dots \dot{\alpha}_{2s+1}} e^{ip \cdot X} , \quad (26)$$

$$Q_2 \widetilde{W}^{\dot{\alpha}_1 \dots \dot{\alpha}_{2s}} \theta^\alpha e^{ip \cdot X} = ip^{\alpha(\dot{\alpha}_1} \widetilde{W}^{\dot{\alpha}_2 \dots \dot{\alpha}_{2s})} e^{ip \cdot X} . \quad (27)$$

A factor of γ or $\partial\gamma$ may be included on both sides of the equation in either formula. Note that in (27), the right-hand side is defined to be zero if $s = 0$. It is worth remarking that we have recovered the manifest covariance under $SL(2, R)_L$ in the expressions for the $W^{\dot{\alpha}_1 \dots \dot{\alpha}_{2s}}$ and $\widetilde{W}^{\dot{\alpha}_1 \dots \dot{\alpha}_{2s}}$, even though it was broken by the bosonisation of the ghosts.

3.2 Supersymmetry transformations

The $N = 1$ spacetime supersymmetry provides an organising principle for the physical states. It is therefore convenient to begin by writing down a subset of the physical states that can then be filled out into multiplets by using the supersymmetry generators. We shall focus here on such a subset for a class of massless states in the theory. They are described by the following operators:

$$U = h_{\alpha \dot{\alpha}_1 \dots \dot{\alpha}_{2s}} c \gamma W^{\dot{\alpha}_1 \dots \dot{\alpha}_{2s}} \theta^\alpha e^{ip \cdot X} , \quad (28)$$

$$\widetilde{U} = \tilde{h}_{\alpha \dot{\alpha}_1 \dots \dot{\alpha}_{2s}} c \gamma \widetilde{W}^{\dot{\alpha}_1 \dots \dot{\alpha}_{2s}} \theta^\alpha e^{ip \cdot X} , \quad (29)$$

For now, we shall derive the supermultiplets associated with these operators without yet imposing any physical-state conditions. Having then derived the corresponding off-shell supersymmetry transformations in this section, we shall then, in the next section, discuss the physical-state conditions and the associated on-shell transformation rules.

Let us first consider the operator U given in (28). Acting with the supersymmetry generators $\epsilon_\alpha q^\alpha$ and $\epsilon_{\dot{\alpha}} q^{\dot{\alpha}}$, we obtain the complete set of six operators,

$$\begin{aligned} U &= h_{\alpha \dot{\alpha}_1 \dots \dot{\alpha}_{2s}} c \gamma W^{\dot{\alpha}_1 \dots \dot{\alpha}_{2s}} \theta^\alpha e^{ip \cdot X} , \\ V &= g_{\alpha \dot{\alpha}_1 \dots \dot{\alpha}_{2s}} c \partial\gamma \gamma W^{\dot{\alpha}_1 \dots \dot{\alpha}_{2s}} \theta^\alpha e^{ip \cdot X} , \\ \Psi &= h_{\dot{\alpha}_1 \dots \dot{\alpha}_{2s}} c \gamma W^{\dot{\alpha}_1 \dots \dot{\alpha}_{2s}} e^{ip \cdot X} , \\ \Phi &= g_{\dot{\alpha}_1 \dots \dot{\alpha}_{2s}} c \partial\gamma \gamma W^{\dot{\alpha}_1 \dots \dot{\alpha}_{2s}} e^{ip \cdot X} , \\ R &= b_{\dot{\alpha}_1 \dots \dot{\alpha}_{2s}} c \gamma W^{\dot{\alpha}_1 \dots \dot{\alpha}_{2s}} \theta^2 e^{ip \cdot X} , \\ S &= d_{\dot{\alpha}_1 \dots \dot{\alpha}_{2s}} c \partial\gamma \gamma W^{\dot{\alpha}_1 \dots \dot{\alpha}_{2s}} \theta^2 e^{ip \cdot X} , \end{aligned} \quad (30)$$

The action of the supersymmetry generators can then be written in terms of this basis of operators. For example, acting on U with $\epsilon_\alpha q^\alpha$, we find

$$\epsilon_\alpha q^\alpha U = \epsilon^\alpha h_{\alpha \dot{\alpha}_1 \dots \dot{\alpha}_{2s}} c \gamma W^{\dot{\alpha}_1 \dots \dot{\alpha}_{2s}} e^{ip \cdot X} , \quad (31)$$

which can be recognised as an operator of the form Ψ . A similar pattern can be obtained for the remaining supersymmetry transformations of the operators. If we associate spacetime fields

with the polarisation spinors according to the scheme

$$\begin{aligned}
A_{\alpha\dot{\alpha}_1\cdots\dot{\alpha}_{2s}} &\leftrightarrow h_{\alpha\dot{\alpha}_1\cdots\dot{\alpha}_{2s}} e^{ip\cdot X}, & B_{\alpha\dot{\alpha}_1\cdots\dot{\alpha}_{2s}} &\leftrightarrow g_{\alpha\dot{\alpha}_1\cdots\dot{\alpha}_{2s}} e^{ip\cdot X}, \\
\psi_{\dot{\alpha}_1\cdots\dot{\alpha}_{2s}} &\leftrightarrow h_{\dot{\alpha}_1\cdots\dot{\alpha}_{2s}} e^{ip\cdot X}, & \phi_{\dot{\alpha}_1\cdots\dot{\alpha}_{2s}} &\leftrightarrow g_{\dot{\alpha}_1\cdots\dot{\alpha}_{2s}} e^{ip\cdot X}, \\
\rho_{\dot{\alpha}_1\cdots\dot{\alpha}_{2s}} &\leftrightarrow b_{\dot{\alpha}_1\cdots\dot{\alpha}_{2s}} e^{ip\cdot X}, & \sigma_{\dot{\alpha}_1\cdots\dot{\alpha}_{2s}} &\leftrightarrow d_{\dot{\alpha}_1\cdots\dot{\alpha}_{2s}} e^{ip\cdot X},
\end{aligned} \tag{32}$$

we can obtain the supersymmetry algebra of the spacetime fields. For example, the transformation given in (31) gives rise to $\delta\psi_{\dot{\alpha}_1\cdots\dot{\alpha}_{2s}} = \epsilon^\alpha A_{\alpha\dot{\alpha}_1\cdots\dot{\alpha}_{2s}}$. The complete set of such supersymmetry transformations takes the form

$$\begin{aligned}
\delta A_{\alpha\dot{\alpha}_1\cdots\dot{\alpha}_{2s}} &= \epsilon^{\dot{\alpha}} \partial_{\alpha\dot{\alpha}} \psi_{\dot{\alpha}_1\cdots\dot{\alpha}_{2s}} + \epsilon_\alpha \rho_{\dot{\alpha}_1\cdots\dot{\alpha}_{2s}}, \\
\delta\psi_{\dot{\alpha}_1\cdots\dot{\alpha}_{2s}} &= \epsilon^\alpha A_{\alpha\dot{\alpha}_1\cdots\dot{\alpha}_{2s}}, \\
\delta\rho_{\dot{\alpha}_1\cdots\dot{\alpha}_{2s}} &= \epsilon_{\dot{\alpha}} \partial^{\alpha\dot{\alpha}} A_{\alpha\dot{\alpha}_1\cdots\dot{\alpha}_{2s}}, \\
\delta B_{\alpha\dot{\alpha}_1\cdots\dot{\alpha}_{2s-1}} &= -\epsilon^{\dot{\alpha}_{2s}} A_{\alpha\dot{\alpha}_1\cdots\dot{\alpha}_{2s}} + \epsilon^{\dot{\alpha}} \partial_{\alpha\dot{\alpha}} \phi_{\dot{\alpha}_1\cdots\dot{\alpha}_{2s-1}} + \epsilon_\alpha \sigma_{\dot{\alpha}_1\cdots\dot{\alpha}_{2s-1}}, \\
\delta\phi_{\dot{\alpha}_1\cdots\dot{\alpha}_{2s-1}} &= \epsilon^{\dot{\alpha}_{2s}} \psi_{\dot{\alpha}_1\cdots\dot{\alpha}_{2s}} + \epsilon^\alpha B_{\alpha\dot{\alpha}_1\cdots\dot{\alpha}_{2s-1}}, \\
\delta\sigma_{\dot{\alpha}_1\cdots\dot{\alpha}_{2s-1}} &= \epsilon^{\dot{\alpha}_{2s}} \rho_{\dot{\alpha}_1\cdots\dot{\alpha}_{2s}} + \epsilon_{\dot{\alpha}} \partial^{\alpha\dot{\alpha}} B_{\alpha\dot{\alpha}_1\cdots\dot{\alpha}_{2s-1}}.
\end{aligned} \tag{33}$$

One can easily verify that these transformations close off-shell on the usual $N = 1$ supersymmetry algebra. In the next subsection, we shall show how this algebra truncates to an on-shell algebra of the physical fields, when we impose the physical-state conditions.

We now discuss the supermultiplet associated with the operator \tilde{U} given in (29). In an analogous manner to the above, we obtain five additional operators by acting with the supersymmetry generators. Thus in all we have

$$\begin{aligned}
\tilde{U} &= \tilde{h}_{\alpha\dot{\alpha}_1\cdots\dot{\alpha}_{2s}} c\gamma \tilde{W}^{\dot{\alpha}_1\cdots\dot{\alpha}_{2s}} \theta^\alpha e^{ip\cdot X}, \\
\tilde{V} &= \tilde{g}_{\alpha\dot{\alpha}_1\cdots\dot{\alpha}_{2s}} c\partial\gamma\gamma \tilde{W}^{\dot{\alpha}_1\cdots\dot{\alpha}_{2s}} \theta^\alpha e^{ip\cdot X}, \\
\tilde{\Psi} &= \tilde{h}_{\dot{\alpha}_1\cdots\dot{\alpha}_{2s}} c\gamma \tilde{W}^{\dot{\alpha}_1\cdots\dot{\alpha}_{2s}} \theta^2 e^{ip\cdot X}, \\
\tilde{\Phi} &= \tilde{g}_{\dot{\alpha}_1\cdots\dot{\alpha}_{2s}} c\partial\gamma\gamma \tilde{W}^{\dot{\alpha}_1\cdots\dot{\alpha}_{2s}} \theta^2 e^{ip\cdot X}, \\
\tilde{R} &= \tilde{b}_{\dot{\alpha}_1\cdots\dot{\alpha}_{2s}} c\gamma \tilde{W}^{\dot{\alpha}_1\cdots\dot{\alpha}_{2s}} e^{ip\cdot X}, \\
\tilde{S} &= \tilde{d}_{\dot{\alpha}_1\cdots\dot{\alpha}_{2s}} c\partial\gamma\gamma \tilde{W}^{\dot{\alpha}_1\cdots\dot{\alpha}_{2s}} e^{ip\cdot X},
\end{aligned} \tag{34}$$

We associate spacetime fields with the polarisation spinors precisely as in (32), except that now all quantities carry tildes. The supersymmetry transformation rules are given by

$$\begin{aligned}
\delta\tilde{A}_{\alpha\dot{\alpha}_1\cdots\dot{\alpha}_{2s}} &= \epsilon^{\dot{\alpha}} \partial_{\alpha\dot{\alpha}} \tilde{\rho}_{\dot{\alpha}_1\cdots\dot{\alpha}_{2s}} + \epsilon_\alpha \tilde{\psi}_{\dot{\alpha}_1\cdots\dot{\alpha}_{2s}}, \\
\delta\tilde{\psi}_{\dot{\alpha}_1\cdots\dot{\alpha}_{2s}} &= \epsilon_{\dot{\alpha}} \partial^{\alpha\dot{\alpha}} \tilde{A}_{\alpha\dot{\alpha}_1\cdots\dot{\alpha}_{2s}}, \\
\delta\tilde{\rho}_{\dot{\alpha}_1\cdots\dot{\alpha}_{2s}} &= \epsilon^\alpha \tilde{A}_{\alpha\dot{\alpha}_1\cdots\dot{\alpha}_{2s}}, \\
\delta\tilde{B}^{\dot{\alpha}}_{\dot{\alpha}_1\cdots\dot{\alpha}_{2s+1}} &= -\epsilon_{(\dot{\alpha}_{2s+1}} \tilde{A}^{\dot{\alpha}}_{\dot{\alpha}_1\cdots\dot{\alpha}_{2s}} - \epsilon_{\dot{\alpha}} \partial^{\alpha\dot{\alpha}} \tilde{\sigma}_{\dot{\alpha}_1\cdots\dot{\alpha}_{2s+1}} + \epsilon^\alpha \tilde{\phi}_{\dot{\alpha}_1\cdots\dot{\alpha}_{2s+1}}, \\
\delta\tilde{\phi}_{\dot{\alpha}_1\cdots\dot{\alpha}_{2s+1}} &= \epsilon_{(\dot{\alpha}_{2s+1}} \tilde{\psi}_{\dot{\alpha}_1\cdots\dot{\alpha}_{2s}} + \epsilon^{\dot{\alpha}} \partial_{\alpha\dot{\alpha}} \tilde{B}^{\dot{\alpha}}_{\dot{\alpha}_1\cdots\dot{\alpha}_{2s+1}}, \\
\delta\tilde{\sigma}_{\dot{\alpha}_1\cdots\dot{\alpha}_{2s+1}} &= \epsilon_{(\dot{\alpha}_{2s+1}} \tilde{\rho}_{\dot{\alpha}_1\cdots\dot{\alpha}_{2s}} - \epsilon_{\dot{\alpha}} \tilde{B}^{\dot{\alpha}}_{\dot{\alpha}_1\cdots\dot{\alpha}_{2s+1}}.
\end{aligned} \tag{35}$$

3.3 Physical States

Having established the off-shell structure of the supermultiplets for the operators U and \tilde{U} in (28, 29), we turn to an analysis of the physical-state conditions for these operators. There

are two parts to this analysis; first requiring that the operators be annihilated by the BRST operator, and then investigating the conditions under which they are BRST non-trivial.

Beginning with the untilded operators, we find that U itself is annihilated by Q provided that the following conditions hold:

$$p^{\alpha\dot{\alpha}} p_{\alpha\dot{\alpha}} = 0 , \quad h^{\alpha(\dot{\alpha}_1 \dots \dot{\alpha}_{2s}} p_{\alpha}^{\dot{\alpha})} = 0 . \quad (36)$$

The first of these is just the mass-shell condition for massless states. Having ensured that U is annihilated by Q , we must also check to see whether it is BRST non-trivial. One way to do this is by constructing conjugate operators that have a non-vanishing inner product with U , as defined by (21). If the inner-product is non-vanishing for conjugate operators that are annihilated by Q , then U is BRST non-trivial. Operators U^\dagger conjugate to U have the form

$$U^\dagger = f_{\alpha\dot{\alpha}_1 \dots \dot{\alpha}_{2s}} \partial c c \partial \gamma \gamma \widetilde{W}^{\dot{\alpha}_1 \dots \dot{\alpha}_{2s}} \theta^\alpha , e^{ip \cdot X} , \quad (37)$$

which is annihilated by Q if

$$p^{\alpha\dot{\alpha}_1} f_{\dot{\alpha}_1 \dots \dot{\alpha}_{2s}} = 0 . \quad (38)$$

It is convenient to choose a particular momentum frame in order to analyse the true physical degrees of freedom that are implied by these kinematical conditions. The null momentum vector p^μ may, without loss of generality, be chosen to be $p^\mu = (1, 0, 0, 1)$. From (2), this implies that all components of $p^{\alpha\dot{\alpha}}$ are zero except for $p^{1\dot{1}} = \sqrt{2}$. In this frame, the solutions to (36) and (38) are

$$h_{1\dot{\alpha}_1 \dots \dot{\alpha}_{2s}} = 0 , \quad f_{1\dot{1}\dot{\alpha}_2 \dots \dot{\alpha}_{2s}} = 0 . \quad (39)$$

The inner product has the form $\langle U^\dagger U \rangle = f^{\alpha\dot{\alpha}_1 \dots \dot{\alpha}_{2s}} h_{\alpha\dot{\alpha}_1 \dots \dot{\alpha}_{2s}} = f^{2\dot{1} \dots \dot{1}} h_{2\dot{1} \dots \dot{1}}$. Thus there is just one physical degree of freedom described by U , corresponding to the polarisation spinor component $h_{2\dot{1} \dots \dot{1}}$. The other non-vanishing components of $h_{\alpha\dot{\alpha}_1 \dots \dot{\alpha}_{2s}}$ allowed by (39) correspond to BRST trivial states, and can be expressed back in covariant language as pure-gauge states with

$$h_{\alpha\dot{\alpha}_1 \dots \dot{\alpha}_{2s}} = p_{\alpha(\dot{\alpha}_1} \Lambda_{\dot{\alpha}_2 \dots \dot{\alpha}_{2s})} , \quad (40)$$

where $\Lambda_{\dot{\alpha}_2 \dots \dot{\alpha}_{2s}}$ is arbitrary. We note also, for future reference, that the equation of motion for $h^{\alpha\dot{\alpha}_1 \dots \dot{\alpha}_{2s}}$ in (36) is equivalent to

$$h^{\alpha\dot{\alpha}_1 \dots \dot{\alpha}_{2s}} p_{\alpha}^{\dot{\alpha}} = 0 . \quad (41)$$

The operator Ψ in (30) is annihilated by Q provided just that the mass-shell condition $p^{\alpha\dot{\alpha}} p_{\alpha\dot{\alpha}} = 0$ is satisfied. To see the physical degrees of freedom, we again consider conjugate operators Ψ^\dagger , which have the form $\Psi^\dagger = f_{\dot{\alpha}_1 \dots \dot{\alpha}_{2s}} \partial c c \partial \gamma \gamma \widetilde{W}^{\dot{\alpha}_1 \dots \dot{\alpha}_{2s}} \theta^2 e^{ip \cdot X}$. This is annihilated by Q provided that $p^{\alpha\dot{\alpha}_1} f_{\dot{\alpha}_1 \dots \dot{\alpha}_{2s}} = 0$. In the special momentum frame, the solution is $f_{1\dot{\alpha}_2 \dots \dot{\alpha}_{2s}} = 0$. Thus the inner product is proportional to $h^{2 \dots 2} f_{2 \dots 2}$, so only the one component $h^{2 \dots 2}$ describes a true physical degree of freedom. The unphysical BRST-trivial components correspond to polarisation spinors of the pure-gauge form

$$h^{\dot{\alpha}_1 \dots \dot{\alpha}_{2s}} = p^{\alpha(\dot{\alpha}_1} \Lambda_{\alpha}^{\dot{\alpha}_2 \dots \dot{\alpha}_{2s})} . \quad (42)$$

The analysis of the operators V and Φ in (30) is precisely the same as the above analyses for U and Ψ respectively. The operators R and S in (30) are annihilated by Q only if $b_{\dot{\alpha}_1 \dots \dot{\alpha}_{2s}} = 0$ and $d_{\dot{\alpha}_1 \dots \dot{\alpha}_{2s}} = 0$, and thus they do not describe any physical degrees of freedom. The

corresponding spacetime fields $\rho_{\dot{\alpha}_1 \dots \dot{\alpha}_{2s}}$ and $\sigma_{\dot{\alpha}_1 \dots \dot{\alpha}_{2s}}$ can be thought of as auxiliary fields that are needed in order to have a supersymmetry algebra (33) that closes off-shell. In fact one can easily see in the off-shell transformations (33) that it is consistent to set $\rho_{\dot{\alpha}_1 \dots \dot{\alpha}_{2s}}$ and $\sigma_{\dot{\alpha}_1 \dots \dot{\alpha}_{2s}}$ to zero, since the right-hand sides of their variations then vanish by virtue of the field equations for $A_{\alpha \dot{\alpha}_1 \dots \dot{\alpha}_{2s}}$ and $B_{\alpha \dot{\alpha}_1 \dots \dot{\alpha}_{2s-1}}$, which can be read off from (41). The on-shell supersymmetry transformations are thus given by the remaining four variations in (33), with $\rho_{\dot{\alpha}_1 \dots \dot{\alpha}_{2s}}$ and $\sigma_{\dot{\alpha}_1 \dots \dot{\alpha}_{2s}}$ set to zero, namely

$$\begin{aligned}\delta A_{\alpha \dot{\alpha}_1 \dots \dot{\alpha}_{2s}} &= \epsilon^{\dot{\alpha}} \partial_{\alpha \dot{\alpha}} \psi_{\dot{\alpha}_1 \dots \dot{\alpha}_{2s}} , \\ \delta \psi_{\dot{\alpha}_1 \dots \dot{\alpha}_{2s}} &= \epsilon^{\alpha} A_{\alpha \dot{\alpha}_1 \dots \dot{\alpha}_{2s}} , \\ \delta B_{\alpha \dot{\alpha}_1 \dots \dot{\alpha}_{2s-1}} &= -\epsilon^{\dot{\alpha}_{2s}} A_{\alpha \dot{\alpha}_1 \dots \dot{\alpha}_{2s}} + \epsilon^{\dot{\alpha}} \partial_{\alpha \dot{\alpha}} \phi_{\dot{\alpha}_1 \dots \dot{\alpha}_{2s-1}} , \\ \delta \phi_{\dot{\alpha}_1 \dots \dot{\alpha}_{2s-1}} &= \epsilon^{\dot{\alpha}_{2s}} \psi_{\dot{\alpha}_1 \dots \dot{\alpha}_{2s}} + \epsilon^{\alpha} B_{\alpha \dot{\alpha}_1 \dots \dot{\alpha}_{2s-1}} .\end{aligned}\tag{43}$$

In the special momentum frame that we discussed above, the physical degrees of freedom are described by the components $A_{2\dot{1}\dots\dot{1}}$, $B_{2\dot{1}\dots\dot{1}}$, $\psi_{\dot{1}\dots\dot{1}}$ and $\phi_{\dot{1}\dots\dot{1}}$ of the spacetime fields.

It is instructive to examine the supermultiplet in more detail in the special case $s = \frac{1}{2}$. The on-shell supersymmetry transformations now take the form

$$\begin{aligned}\delta A_{\alpha \dot{\alpha}} &= \epsilon^{\dot{\beta}} \partial_{\alpha \dot{\beta}} \psi_{\dot{\alpha}} , \\ \delta \psi_{\dot{\alpha}} &= \epsilon^{\alpha} A_{\alpha \dot{\alpha}} , \\ \delta B_{\alpha} &= -\epsilon^{\dot{\alpha}} A_{\alpha \dot{\alpha}} + \epsilon^{\dot{\alpha}} \partial_{\alpha \dot{\alpha}} \phi , \\ \delta \phi &= \epsilon^{\dot{\alpha}} \psi_{\dot{\alpha}} + \epsilon^{\alpha} B_{\alpha} .\end{aligned}\tag{44}$$

The multiplet is reducible. First, we note that $\{A_{\alpha \dot{\alpha}}, \psi_{\dot{\alpha}}\}$ satisfy a closed algebra of spins $\{(\frac{1}{2}, \frac{1}{2}), (0, \frac{1}{2})\}$. If instead we set $A_{\alpha \dot{\alpha}}$ and $\psi_{\dot{\alpha}}$ to zero, we get an irreducible multiplet $\{B_{\alpha}, \phi\}$ of spins $\{(\frac{1}{2}, 0), (0, 0)\}$. A third irreducible multiplet can be obtained by setting instead $B_{\alpha} = 0$, which implies that $A_{\alpha \dot{\alpha}} = \partial_{\alpha \dot{\alpha}} \phi$. This corresponds to a multiplet $\{\psi_{\dot{\alpha}}, \phi\}$ with spins $\{(0, \frac{1}{2}), (0, 0)\}$. A fourth irreducible multiplet can be obtained by defining $\hat{A}_{\alpha \dot{\alpha}} = A_{\alpha \dot{\alpha}} - \partial_{\alpha \dot{\alpha}} \phi$. This corresponds to a multiplet $\{\hat{A}_{\alpha \dot{\alpha}}, B_{\alpha}\}$ of spins $\{(\frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, 0)\}$. This reducibility of the supermultiplet occurs for all values of s , giving rise to analogous irreducible multiplets of spins $\{(\frac{1}{2}, s), (0, s)\}$, $\{(\frac{1}{2}, s - \frac{1}{2}), (0, s - \frac{1}{2})\}$, $\{(0, s), (0, s - \frac{1}{2})\}$ and $\{(\frac{1}{2}, s), (\frac{1}{2}, s - \frac{1}{2})\}$ respectively.

The field equation for $A_{\alpha \dot{\alpha}}$ in the $s = \frac{1}{2}$ case, which follows from the physical-state condition (36), is $\partial^{\alpha(\dot{\alpha}} A_{\alpha}^{\dot{\beta})} = 0$. This is invariant under gauge transformation $A_{\alpha \dot{\alpha}} \rightarrow A_{\alpha \dot{\alpha}} + \partial_{\alpha \dot{\alpha}} \Lambda$. Thus the gauge-invariant field strength $F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu} = F_{\alpha\beta} \epsilon_{\dot{\alpha}\dot{\beta}} + F_{\dot{\alpha}\dot{\beta}} \epsilon_{\alpha\beta}$ has $F_{\dot{\alpha}\dot{\beta}} \equiv \partial_{\alpha(\dot{\alpha}} A_{\alpha}^{\dot{\beta})} = 0$, whilst $F_{\alpha\beta} \equiv -\partial_{(\alpha}^{\dot{\alpha}} A_{\beta)}^{\dot{\alpha}}$ is non-zero. It therefore corresponds to a self-dual gauge field. Note that the superpartner $\psi_{\dot{\alpha}}$ is left-handed, and consequently they form a non-standard realisation of the supersymmetry algebra, as given in (44). The unusual dimension for the spinor field $\psi_{\dot{\alpha}}$ is related to the fact that it does not satisfy the usual Dirac equation. However, we can define a new spinor field $\chi_{\alpha} = \partial_{\alpha \dot{\alpha}} \psi^{\dot{\alpha}}$, which does satisfy the Dirac equation. In the special momentum frame, the physical degree of freedom is carried by the component χ_2 . In terms of χ_{α} , the supersymmetry transformations for the irreducible multiplet can be written in the standard form

$$\begin{aligned}\delta A_{\alpha \dot{\alpha}} &= \epsilon_{\dot{\alpha}} \chi_{\alpha} , \\ \delta \chi_{\alpha} &= \epsilon^{\dot{\beta}} F_{\alpha \dot{\beta}} .\end{aligned}\tag{45}$$

In obtaining this transformation rule from (44), we have dropped pure gauge terms, and made use of the physical-state conditions.

Similar redefinitions can be performed for higher values of s . In this case, we define $C_{\alpha_1 \dots \alpha_{2s} \dot{\alpha}} = \partial_{(\alpha_2}^{\dot{\alpha}_2} \dots \partial_{\alpha_{2s})}^{\dot{\alpha}_{2s}} A_{\alpha_1 \dot{\alpha}_2 \dots \dot{\alpha}_{2s} \dot{\alpha}}$, and $\chi_{\alpha_1 \dots \alpha_{2s}} = \partial_{\alpha_1}^{\dot{\alpha}_1} \dots \partial_{\alpha_{2s}}^{\dot{\alpha}_{2s}} \psi_{\dot{\alpha}_1 \dots \dot{\alpha}_{2s}}$. The transformation rules become, after dropping pure-gauge terms and using the physical-state conditions,

$$\begin{aligned} \delta C_{\alpha_1 \dots \alpha_{2s} \dot{\alpha}} &= \epsilon_{\dot{\alpha}} \chi_{\alpha_1 \dots \alpha_{2s}} , \\ \delta \chi_{\alpha_1 \dots \alpha_{2s}} &= \epsilon^{\alpha_{2s+1}} F_{\alpha_1 \dots \alpha_{2s+1}} , \end{aligned} \quad (46)$$

where $F_{\alpha_1 \dots \alpha_{2s+1}} \equiv \partial_{(\alpha_1}^{\dot{\alpha}} C_{\alpha_2 \dots \alpha_{2s+1}) \dot{\alpha}}$ is a generalised self-dual field strength. It is worth remarking that even though the redefinitions involve derivatives of the original fields, the on-shell degrees of freedom are preserved, as can be easily verified by using the special momentum frame.

At this point, a comment is in order. The supersymmetry transformation rules given in (43) and (46) close on-shell for all values of s . This implies that sometimes a boson is transformed into the derivative of a fermion. However, the unusual dimensions of the fields implied by this are not inconsistent; they reflect the fact that one member of the supermultiplet does not satisfy a conventional equation of motion. The original field $A_{\alpha \dot{\alpha}_1 \dots \dot{\alpha}_{2s}}$ satisfies the field equation implied by (36),

$$\partial_{\alpha(\dot{\alpha}} A^{\alpha}_{\dot{\alpha}_1 \dots \dot{\alpha}_{2s})} = 0 . \quad (47)$$

The original field $\psi_{\dot{\alpha}_1 \dots \dot{\alpha}_{2s}}$ does not have a covariant first order field equation. These original fields have gauge invariances implied by (40) and (42),

$$\delta A_{\alpha \dot{\alpha}_1 \dots \dot{\alpha}_{2s}} = \partial_{\alpha(\dot{\alpha}_1} \Lambda_{\dot{\alpha}_2 \dots \dot{\alpha}_{2s})} , \quad \delta \psi_{\dot{\alpha}_1 \dots \dot{\alpha}_{2s}} = \partial_{\alpha(\dot{\alpha}_1} \Lambda^{\alpha}_{\dot{\alpha}_2 \dots \dot{\alpha}_{2s})} . \quad (48)$$

The situation is reversed for the redefined fields. The field $C_{\alpha_1 \dots \alpha_{2s} \dot{\alpha}}$ does not have a covariant first order field equation whilst the field $\chi_{\alpha_1 \dots \alpha_{2s}}$ satisfies the Dirac-type equation

$$\partial^{\alpha_1 \dot{\alpha}} \chi_{\alpha_1 \dots \alpha_{2s}} = 0 . \quad (49)$$

There is no gauge symmetry for the $\chi_{\alpha_1 \dots \alpha_{2s}}$ field. In fact the equation of motion implies that there is only one on-shell degree of freedom. In the special momentum frame, it is $\chi_{2\dots 2}$. The field $C_{\alpha_1 \dots \alpha_{2s} \dot{\alpha}}$ has the gauge invariance

$$\delta C_{\alpha_1 \dots \alpha_{2s} \dot{\alpha}} = \partial_{(\alpha_1 | \dot{\alpha}} \lambda_{\alpha_2 \dots \alpha_{2s})} . \quad (50)$$

The situation for the $B_{\alpha \dot{\alpha}_1 \dots \dot{\alpha}_{2s-1}}$ and $\phi_{\dot{\alpha}_1 \dots \dot{\alpha}_{2s-1}}$ fields is completely analogous.

We now turn to the tilded physical operators of the forms given by (34). The operators \tilde{R} and \tilde{S} are annihilated by Q provided simply that the mass-shell condition is satisfied. However, these operators are BRST trivial, which can be seen by showing that there exist no conjugate operators that are annihilated by Q . The operator \tilde{V} is annihilated by Q provided that the mass-shell condition is satisfied and that the polarisation spinor satisfies

$$p^{\alpha \dot{\alpha}_1} \tilde{g}_{\alpha \dot{\alpha}_1 \dots \dot{\alpha}_{2s}} = 0 . \quad (51)$$

Similarly, the polarisation spinor in the operator $\tilde{\Phi}$ must satisfy

$$p^{\alpha \dot{\alpha}_1} \tilde{g}_{\dot{\alpha}_1 \dots \dot{\alpha}_{2s}} = 0 . \quad (52)$$

In the special momentum frame introduced earlier, this condition implies that only the component $\tilde{g}_{\dot{2}\dots\dot{2}}$ is non-zero. By considering operators conjugate to \tilde{V} , it is easy to see that only the component $\tilde{g}_{1\dot{2}\dots\dot{2}}$ in \tilde{V} describes a BRST non-trivial physical state.

The physical-state condition for \tilde{U} is similar to that for \tilde{V} , and the only BRST non-trivial physical state is given by the $\tilde{h}_{1\dot{2}\dots\dot{2}}$ component. However, the analysis for $\tilde{\Psi}$ is more complicated. By itself, $\tilde{\Psi}$ can never be annihilated by Q , but together with an additional term, $\tilde{\Psi}$ can become physical. The form of the new operator is

$$\hat{\Psi} = \tilde{\Psi} + \tilde{t}_{\alpha\dot{\alpha}_1\dots\dot{\alpha}_{2s+1}} c \partial\gamma \gamma \tilde{W}^{\dot{\alpha}_1\dots\dot{\alpha}_{2s+1}} \theta^\alpha e^{ip\cdot X} . \quad (53)$$

Note that the additional term is of precisely the same off-shell form as \tilde{V} , so introducing this term does not affect the discussion of the supersymmetry transformation rules (35). On shell, however, the polarisation spinor $\tilde{t}_{\alpha\dot{\alpha}_1\dots\dot{\alpha}_{2s+1}}$ does not satisfy the same condition as $\tilde{g}_{\alpha\dot{\alpha}_1\dots\dot{\alpha}_{2s+1}}$. The physical-state condition for $\hat{\Psi}$ implies

$$\begin{aligned} p^{\alpha\dot{\alpha}_1} \tilde{h}_{\dot{\alpha}_1\dots\dot{\alpha}_{2s}} &= 0 , \\ \tilde{h}_{\dot{\alpha}_1\dots\dot{\alpha}_{2s}} &= p^{\alpha\dot{\alpha}_{2s+1}} \tilde{t}_{\alpha\dot{\alpha}_1\dots\dot{\alpha}_{2s+1}} . \end{aligned} \quad (54)$$

In the special momentum frame, this new BRST non-trivial state is described by the component $\tilde{h}_{\dot{2}\dots\dot{2}} = \sqrt{2} \tilde{t}_{1\dot{1}\dot{2}\dots\dot{2}}$. Thus the complete on-shell multiplet of tilded fields comprises $\tilde{B}_{1\dot{2}\dot{2}\dots\dot{2}}$, $\tilde{\phi}_{\dot{2}\dot{2}\dots\dot{2}}$, $\tilde{A}_{1\dot{2}\dots\dot{2}}$ and $\tilde{\psi}_{\dot{2}\dots\dot{2}}$. The supersymmetry transformation rules for these fields can be read off from (35). Reverting to covariant notation, they take the form

$$\begin{aligned} \delta \tilde{A}_{\alpha\dot{\alpha}_1\dots\dot{\alpha}_{2s}} &= \epsilon_\alpha \tilde{\psi}_{\dot{\alpha}_1\dots\dot{\alpha}_{2s}} , \\ \delta \tilde{\psi}_{\dot{\alpha}_1\dots\dot{\alpha}_{2s}} &= \epsilon_{\dot{\alpha}} \partial^{\alpha\dot{\alpha}} \tilde{A}_{\alpha\dot{\alpha}_1\dots\dot{\alpha}_{2s}} , \\ \delta \tilde{B}^\alpha_{\dot{\alpha}_1\dots\dot{\alpha}_{2s+1}} &= -\epsilon_{(\dot{\alpha}_{2s+1}} \tilde{A}^\alpha_{\dot{\alpha}_1\dots\dot{\alpha}_{2s})} + \epsilon^\alpha \tilde{\phi}_{\dot{\alpha}_1\dots\dot{\alpha}_{2s+1}} , \\ \delta \tilde{\phi}_{\dot{\alpha}_1\dots\dot{\alpha}_{2s+1}} &= \epsilon_{(\dot{\alpha}_{2s+1}} \tilde{\psi}_{\dot{\alpha}_1\dots\dot{\alpha}_{2s})} + \epsilon^{\dot{\alpha}} \partial_{\alpha\dot{\alpha}} \tilde{B}^\alpha_{\dot{\alpha}_1\dots\dot{\alpha}_{2s+1}} . \end{aligned} \quad (55)$$

The situation here is different from that for the untilded on-shell transformations in that pure-gauge terms have been introduced by re-writing the on-shell transformations in a covariant form. Consequently, the on-shell algebra closes only modulo pure-gauge terms, in addition to the use of the field equations. The reason for this difference is that in the untilded case, the physical-state conditions implied that the R and S operators given in (30) were zero. In the tilded case, on the other hand, the physical-state conditions imply instead that the \tilde{R} and \tilde{S} operators are BRST trivial. Thus they describe gauge degrees of freedom. The $\tilde{\psi}_{\dot{\alpha}_1\dots\dot{\alpha}_{2s}}$ field satisfies a Dirac-type field equation:

$$\partial^{\alpha\dot{\alpha}_1} \tilde{\psi}_{\dot{\alpha}_1\dots\dot{\alpha}_{2s}} = 0 . \quad (56)$$

For the same reason that has been explained for the untilded case, the gauge field $\tilde{A}_{\alpha\dot{\alpha}_1\dots\dot{\alpha}_{2s}}$ does not satisfy a covariant field equation. However, the physical-state conditions imply that

$$\begin{aligned} \partial^{\alpha\dot{\alpha}_1} \tilde{A}_{\alpha\dot{\alpha}_1\dots\dot{\alpha}_{2s}} &= 0 , \\ \delta \tilde{A}_{\alpha\dot{\alpha}_1\dots\dot{\alpha}_{2s}} &= \partial_{\alpha}^{\dot{\alpha}_{2s+1}} \tilde{\Lambda}_{\dot{\alpha}_1\dots\dot{\alpha}_{2s+1}} . \end{aligned} \quad (57)$$

The discussion for the fields $\tilde{\phi}_{\dot{\alpha}_1\dots\dot{\alpha}_{2s}}$ and $\tilde{B}_{\alpha\dot{\alpha}_1\dots\dot{\alpha}_{2s}}$ is precisely the same as the above.

The on-shell supersymmetry algebra is reducible. Firstly, we note that $\{\tilde{A}_{\alpha\dot{\alpha}_1\cdots\dot{\alpha}_{2s}}, \tilde{\psi}_{\dot{\alpha}_1\cdots\dot{\alpha}_{2s}}\}$ form an irreducible supermultiplet with spins $\{(\frac{1}{2}, s), (0, s)\}$. If instead we set these two fields to zero, then the fields $\{\tilde{B}_{\alpha\dot{\alpha}_1\cdots\dot{\alpha}_{2s+1}}, \tilde{\phi}_{\dot{\alpha}_1\cdots\dot{\alpha}_{2s+1}}\}$ form an irreducible multiplet with spins $\{(\frac{1}{2}, s + \frac{1}{2}), (0, s + \frac{1}{2})\}$. In the special case $s = \frac{1}{2}$, the former multiplet describes an anti-self-dual vector gauge field and a left-handed spinor. Their supersymmetry transformation rules are

$$\begin{aligned}\delta\tilde{A}_{\alpha\dot{\alpha}} &= \epsilon_{\alpha}\tilde{\psi}_{\dot{\alpha}} , \\ \delta\tilde{\psi}_{\dot{\alpha}} &= \epsilon^{\dot{\beta}}\tilde{F}_{\dot{\alpha}\dot{\beta}} ,\end{aligned}\tag{58}$$

where $\tilde{F}_{\dot{\alpha}\dot{\beta}} = \partial_{\alpha(\dot{\alpha}}\tilde{A}^{\alpha}_{\dot{\beta})}$ is the anti-self-dual field strength. This multiplet generalises for higher values of s , with a generalised anti-self-dual field strength $\tilde{F}_{\dot{\alpha}_1\cdots\dot{\alpha}_{2s+1}} = \partial_{\alpha(\dot{\alpha}_1}\tilde{A}^{\alpha}_{\dot{\alpha}_2\cdots\dot{\alpha}_{2s+1})}$. A similar discussion applies to the other multiplet $\{\tilde{B}_{\alpha\dot{\alpha}_1\cdots\dot{\alpha}_{2s+1}}, \tilde{\phi}_{\dot{\alpha}_1\cdots\dot{\alpha}_{2s+1}}\}$.

So far, we have concentrated on massless states in the physical spectrum. There are also massive physical states, an example being $c e^{-\phi_1-\phi_2} e^{ip\cdot X}$ with $p^{\alpha\dot{\alpha}} p_{\alpha\dot{\alpha}} = -2$, implying $(\text{mass})^2 = 2$. Further examples are

$$V = c \partial^{2n} \beta \cdots \partial \beta \beta (\partial^n p)^2 \cdots (\partial p)^2 p^2 e^{n(\phi_1+\phi_2)} e^{ip\cdot X} ,\tag{59}$$

where $p^2 = p^{\alpha} p_{\alpha}$, *etc.* These spacetime scalar states are physical for arbitrary integer n , with $(\text{mass})^2 = 2(n+1)(2n+3)$.

Another class of physical states in the theory is associated with infinite-dimensional representations of $SL(2, R)_{\text{L}}$. Consider, for example, the operator

$$V = c \partial \gamma \gamma \theta^2 e^{-2\phi_1-\phi_2} e^{ip\cdot X} .\tag{60}$$

This is annihilated by the BRST operator provided that the mass-shell condition $p^{\alpha\dot{\alpha}} p_{\alpha\dot{\alpha}} = 0$ is satisfied, together with the transversality condition $p^{\alpha^2} = 0$. This condition is not covariant with respect to $SL(2, R)_{\text{L}}$, suggesting that further terms should be added in order to construct a fully-covariant physical operator. This is analogous to viewing a physical operator built using $W_{\dot{\alpha}_1\cdots\dot{\alpha}_{2s}}$ as consisting of the term involving $W_{\dot{1}\cdots\dot{1}}$ plus the remaining $2s$ terms obtained by acting repeatedly on this highest-weight state with J_- . Thus, noting that $e^{-2\phi_1-\phi_2}$ is a highest-weight state, $J_+ e^{-2\phi_1-\phi_2} = 0$, we may replace (60) by the $SL(2, R)_{\text{L}}$ covariant operator

$$V = \sum_{n \geq 0} h_n c \partial \gamma \gamma \theta^2 \left((J_-)^n e^{-2\phi_1-\phi_2} \right) e^{ip\cdot X} .\tag{61}$$

One can easily see from the form of the generator J_- in (17) that in this case the process of repeated application of J_- will never terminate, and the sum over n will be an infinite one, corresponding to an infinite-dimensional representation of $SL(2, R)_{\text{L}}$. The physical-state conditions will now give a transversality condition on the components h_n of the polarisation tensor, rather than the non-covariant condition $p^{\alpha^2} = 0$ that resulted when only the $n = 0$ term was included. The occurrence of infinite-dimensional representations of $SL(2, R)_{\text{L}}$ seems to be an undesirable feature of the theory, and one may hope that some sort of a truncation may be possible in which such physical states are projected out of the spectrum. An understanding of this point presumably will depend upon knowing the detailed form of the interactions in the theory.

4 Discussion

In this paper, we have constructed a superstring theory in four-dimensional spacetime with $(2, 2)$ signature, using the Berkovits' approach of augmenting the spacetime supercoordinates by the conjugate momenta for the fermionic variables [2]. The form of the theory, and its local worldsheet symmetries, was motivated by Siegel's proposal [5] for a set of constraints that could give rise to self-dual super Yang-Mills theory or supergravity in $2 + 2$ dimensions. Such a theory might provide one way to generalise the results of Ooguri and Vafa [3], who showed that by starting with an NSR-type string with $N = 2$ worldsheet supersymmetry, one obtains a theory whose physical spectrum describes (purely bosonic) self-dual Yang-Mills or gravity. In the theory that we have considered, $N = 1$ spacetime supersymmetry is manifest in the formulation, as is the right-handed $SL(2, R)$ factor of the $SO(2, 2) \equiv SL(2, R)_L \times SL(2, R)_R$ Lorentz group. Owing to the fact that the fermionic constraint carries an $SL(2, R)_L$ spacetime spinor index, the $SL(2, R)_L$ symmetry of the physical spectrum, although present, is not always manifest. It is however manifest for the massless states of arbitrary spin that we considered in this paper.

The constraints that we have used are a subset of Siegel's constraints [5] that form a closed algebra under commutation. They give rise to a string theory with an infinite number of massless states with arbitrary spin. This feature can be attributed to the fact that the fermionic constraint carries an $SL(2, R)_L$ spinor index, leading to the existence of ghost vacua with arbitrary spin under $SL(2, R)_L$.

A full analysis of the physical spectrum will require an understanding of the interactions of the physical states. There are two conditions to build a non-vanishing n -point function. One is the conservation law of momentum of the physical operators. The other is that the product of the physical operators has to include the structure that gives a non-vanishing inner product, as given in (21). Interactions amongst the physical states that we have found so far are not easy to come by. One example that can occur is a three-point interaction between two fermions and a boson in a scalar supermultiplet corresponding to the $s = 0$ operators $\langle UU\Psi \rangle$, as given in (30). For higher values of s , interactions necessarily involve tilded physical states and picture-changing operators. However all the tilded physical states have vanishing normal-order product with picture-changing operators, and thus we expect that there are no interactions among these states. It is of interest to have a full analysis of the physical spectrum and interactions of the theory.

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